Int J Theor Phys (2010) 49: 3166–3175
DOI 10.1007/s10773-010-0444-9

Order Topology and Frink Ideal Topology of Effect Algebras

Qiang Lei · Junde Wu · Ronglu Li

Received: 13 August 2008 / Accepted: 22 July 2010 / Published online: 5 August 2010

Abstract In this paper, the following results are proved: (1) If E is a complete atomic lattice effect algebra, then E is (o)-continuous iff E is order-topological iff E is totally order-disconnected iff E is algebraic. (2) If E is a complete atomic distributive lattice effect algebra, then its Frink ideal topology τ_{id} is Hausdorff topology and τ_{id} is finer than its order topology τ_o, and $\tau_{id} = \tau_o$ iff 1 is finite iff every element of E is finite iff τ_{id} and τ_o are both discrete topologies. (3) If E is a complete (o)-continuous lattice effect algebra and the operation $\oplus$ is order topology τ_o continuous, then its order topology τ_o is Hausdorff topology. (4) If E is a (o)-continuous complete atomic lattice effect algebra, then $\oplus$ is order topology continuous.

Keywords Effect algebras · Order topology · Frink ideal topology

1 Introduction

Effect algebra is an important model in studying the unsharp quantum logic theory, it is also an important carrier of quantum states and quantum measures [1]. As an important tool of studying the quantum states and quantum measures, the topological structures of effect algebras not only can help us to describe the convergence properties of quantum states and quantum measures, but also can help us to characterize some algebra properties of effect algebras. This paper contributes to the understanding of the topological properties and algebraic properties of effect algebras, it both promotes some classical results, for example,

This project is supported by Natural Science Foundation of China (10771191 and 10471124) and Natural Science Foundation of Zhejiang Province of China (Y6090105).

Q. Lei (✉) · R. Li
Department of Mathematics, Harbin Institute of Technology, Harbin, China
e-mail: leiqiang@hit.edu.cn

J. Wu
Department of Mathematics, Zhejiang University, Hangzhou, China
e-mail: wjd@zju.edu.cn

Theorem 2.1, and obtain several new interesting conclusions, for example, Theorem 3.1, Theorem 4.1 and Theorem 4.3, etc. Now, we show them in details in the following three sections.

The structure $(E, \oplus, 0, 1)$ is said to be an effect algebra if 0, 1 are two distinguished elements of E and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a, b, c \in E$ [1]:

(1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
(2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(3) For each $a \in E$ there exists a unique $b \in E$ such that $a \oplus b$ is defined and $a \oplus b = 1$.
(4) If $a \oplus 1$ is defined, then $a = 0$.

The effect algebra $(E, \oplus, 0, 1)$ is often denoted by E. For every $a \in E$, we denote the unique b in condition (3) by a' and call it the orthosupplement of a. The sense is that if a presents a proposition, then a' corresponds to its negation. The operation $\oplus$ of an effect algebra E can induce a new partial operation $\ominus$ and a partial order $\leq$ as follows: $a \ominus b$ is defined iff there exists $c \in E$ such that $b \oplus c$ is defined and $b \oplus c = a$, in which case we denote c by $a \ominus b$; $d \leq e$ iff there exists $f \in E$ such that $d \oplus f$ is defined and $d \oplus f = e$. This showed that every effect algebra is a partial order set. If $(E, \leq)$ is a lattice, then E is called a lattice effect algebra, similarly, we can define the complete lattice effect algebras. If a lattice effect algebra is a distributive lattice, then it is called a distributive lattice effect algebra. For more details on effect algebras, for example, $a \oplus b$ is defined iff $a \leq b'$, we refer to [1].

Let E be an effect algebra and $a, b \in E$ with $a \leq b$, denote $[a, b] = \{p \in E | a \leq p \leq b\}$. A nonzero element $a \in E$ is said to be an atom of E if $[0, a] = \{0, a\}$, E is said to be atomic iff for every nonzero element $p \in E$, there is an atom $a \in E$ such that $a \leq p$ [1].

An element $x \in E$ is said to be a sharp element of E if $x \wedge x' = 0$, that is, proposition x and its negation x' have no overlaps.

2 The Order-Continuity and Order-Topological Effect Algebras

Assume that P is a partial order set and $(a_\alpha)_{\alpha \in \varepsilon}$ is a net of P. If for any $\alpha, \beta \in \varepsilon$, when $\alpha \preceq \beta$, $a_\alpha \leq a_\beta$, then we denote $a_\alpha \uparrow$, moreover, if $a = \bigvee\{a_\alpha | \alpha \in \varepsilon\}$, then we denote $a_\alpha \uparrow a$. Dual, we have $a_\alpha \downarrow$ and $a_\alpha \downarrow a$. A net $(a_\alpha)_{\alpha \in \varepsilon}$ is said to be order convergent ((o)-convergent, for short) to a point $a \in P$ if there are nets $(u_\alpha)_{\alpha \in \varepsilon}$ and $(v_\alpha)_{\alpha \in \varepsilon}$ of P such that $a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a$, and we denote $a_\alpha \xrightarrow{(o)} a$. If τ is a topology equipped on P such that every (o)-convergent net of P is τ- convergent, then τ is said to have C property. The strongest topology of P which has C property is called the order topology of P and denoted by τ_o. It is obvious that the (o)-convergence of nets implies τ_o-convergence, but the converse does not hold [2]. The following Theorem 2.1 answer when they are equal.

A lattice L is said to be (o)-continuous if $x_\alpha, x, y \in L$ and $x_\alpha \uparrow x$ implies that $x_\alpha \wedge y \uparrow x \wedge y$.

It can be proved that if E is an (o)-continuous lattice effect algebra and $x_\alpha \xrightarrow{(o)} x$ and $y_\alpha \xrightarrow{(o)} y$, then $x_\alpha \vee y_\alpha \xrightarrow{(o)} x \vee y$ and $x_\alpha \wedge y_\alpha \xrightarrow{(o)} x \wedge y$ [2].

A complete lattice effect algebra E is said to be order-topological ((o)-topological) if (o)-convergence of nets of elements coincides with τ_o-convergence and E is (o)-continuous [2].

Lemma 2.1 [2] *A complete atomic* (o)*-continuous lattice effect algebra* E *is* (o)*-topological iff* τ_o *of* E *is Hausdorff.*

A partial order set P is said to be down-directed if every finite subset of P has a lower bound in P.

Lemma 2.2 [3] *A subset* U *of a lattice* L *is open in* τ_o *iff for every directed subset* Y *of* L *and every down-directed subset* Z *of* L *with* $\bigvee Y = \bigwedge Z \in U$, *there exist elements* $y \in Y$ *and* $z \in Z$ *such that* $[y, z]$ *is contained in* U.

An element u of an effect algebra E is called finite if there is a finite sequence $\{p_1, \dots, p_n\}$ of atoms of E such that $u = p_1 \oplus \cdots \oplus p_n$. If E is complete and atomic, then for every $x \in E$ we have $x = \vee\{u \in E | u \leq x, u$ is finite$\}$. Moreover, if E is (o)-continuous, then the join of two finite elements is also finite [4].

An element u of an effect algebra E is called compact if $u \leq \vee D$ for $D \subseteq E$ implies that $u \leq \vee F$ for some finite subset $F \subseteq D$, and E is called algebraic (or compactly generated) if every $x \in E$ is a join of compact elements of E [4].

Lemma 2.3 [4] *Let* E *be a complete atomic* (o)*-continuous lattice effect algebra. Then for every finite element* u *of* E, *if* $u \leq \vee D$ *for* $D \subseteq E$ *implies that* $u \leq \vee F$ *for some finite subset* F *of* D.

Lemma 2.4 *Let* E *be a complete atomic* (o)*-continuous effect algebra. Then for each finite element* u, $[u, 1]$ *and* $[0, u']$ *are* τ_o*-clopen sets.*

Proof Evidently, $[u, 1]$ and $[0, u']$ are τ_o-closed sets. Let Y be a directed subset of E and Z be a down-directed subset of E and satisfy $\bigvee Y = \bigwedge Z \in [u, 1]$. As u is finite, by Lemma 2.3, there exists a subset $\{y_1, \dots, y_n\} \subseteq Y$ such that $u \leq \bigvee_{i=1}^n y_i$. Since Y is directed, there is a $y_0 \in Y$ such that $u \leq \bigvee_{i=1}^n y_i \leq y_0$. For every fixed $z_0 \in Z$, $y_0 \leq \bigvee Y = \bigwedge Z \leq z_0$, so $[y_0, z_0] \subseteq [u, 1]$. By Lemma 2.2, $[u, 1]$ is τ_o-open. □

Similarly, we can prove $[0, u']$ is τ_o-open set.

A lattice L is said to be totally order-disconnected if its lattice operations are order topology τ_o continuous and for any two elements x, y with $x \nleq y$, there exists a clopen upper set U containing x but not y, where U is an upper set iff $u \in U$ implies $\{x \in L : x \geq u\} \subseteq U$.

Lemma 2.5 *Let* E *be a complete atomic* (o)*-continuous lattice effect algebra. Then* E *is totally order-disconnected.*

Proof Let a, b be two elements of E with $a \nleq b$. As $a = \vee\{u \in E | u \leq a, u$ is finite$\}$, there exists a finite element $u_0 \in E$ such that $u_0 \leq a$ and $u_0 \nleq b$. Let $U_1 = [u_0, 1]$. Then $a \in U_1$ and $b \notin U_1$. By Lemma 2.4, U_1 is an upper set and τ_o-clopen. Thus, E is totally order-disconnected. □

It is clear that for every totally order-disconnected lattice, its order topology is disconnected and Hausdorff.

The following theorem establishes the equivalent relation among (o)-continuity, (o)-topological, totally order-disconnected and algebra property of a complete atomic lattice effect algebra E.

Theorem 2.1 *Let E be a complete atomic lattice effect algebra. The following statements are equivalent*:

(1) *E is order-continuous.*
(2) *E is order-topological.*
(3) *E is totally order-disconnected.*
(4) *E is algebraic.*

Proof (1) $\Rightarrow$ (2). It follows from Lemma 2.5 that τ_o is Hausdorff, so by Lemma 2.1, (2) holds. (1) $\Rightarrow$ (3) can be proved by Lemma 2.5 and (1) implies (2). (1) $\Rightarrow$ (4) is obtained by Lemma 2.3. (3) $\Rightarrow$ (2): Since (3) implies that τ_o is Hausdorff and the binary operations of $\wedge$ and $\vee$ are τ_o continuous, by the similar method with the proof of Theorem 8 in [2], (2) holds. For (4) $\Rightarrow$ (1), we refer to [3]. (2) $\Rightarrow$ (1) is clear. □

3 The Relation of Order Topology and Frink Ideal Topology of Effect Algebras

The Frink ideal topology is an important intrinsic topology of partial order set. Frink pointed out that the topology is the correct topology for chains and direct products of a finite numbers of chains. Atherton in [5] asked: Whether the Frink ideal topology is Hausdorff topology in every distributive lattice. Ward pointed out that in a Boolean algebra, the Frink ideal topology is Hausdorff topology and it is quite usual for the Frink ideal topology to be strictly finer than the order topology [6]. Now, we show that for every complete atomic distributive lattice effect algebra E, the Frink ideal topology τ_{id} is Hausdorff and is finer than its order topology τ_o, and $\tau_{id} = \tau_o$ iff 1 is finite iff every element of E is finite iff τ_{id} and τ_o are discrete topologies.

Let L be a lattice and $I \subseteq L$. Then I is said to be an ideal of L if the following conditions are satisfied:

(i) When $a \in I$, $x \in L$ and $x \leq a$, $x \in I$.
(ii) When $a \in I$, $b \in I$, $a \vee b \in I$.

The ideal I of L is said to be a completely irreducible ideal if it is not the intersection of a collection of ideals all distinct from it, i.e., if $(I_\alpha)_{\alpha\in\Lambda}$ is a collection of ideals such that $I = \bigcap_{\alpha\in\Lambda} I_\alpha$, then $I = I_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. It is clear that every maximal ideal is a completely irreducible ideal.

Similarly, the dual ideal and completely irreducible dual ideal of L can be defined, too.

Let L be a lattice. The Frink ideal topology τ_{id} of L can be described as following: Take all completely irreducible ideals and completely irreducible dual ideals of L as a subbasis of the open sets of the topology τ_{id} [7].

Elements a, b of a lattice effect algebra E are said to be compatible iff $a \vee b = a \oplus (b \ominus (a \wedge b))$ and denoted by $a \leftrightarrow b$. If for any $a, b \in E$, $a \leftrightarrow b$, then E is said to be a MV-effect algebra [8].

In order to prove our main results in this section, we first need the following:

Lemma 3.1 [9] *Let E be a lattice effect algebra.*

(i) *If $x \oplus y$ exists, then $x \oplus y = (x \vee y) \oplus (x \wedge y)$.*
(ii) *If $x \wedge y = 0$ and for $m, n \in \mathbf{N}$, the elements mx, ny and $(mx) \oplus (ny)$ exist in E, then $(kx) \wedge (ly) = 0$ and $(kx) \vee (ly) = (kx) \oplus (ly)$ for all $k \in \{1, \ldots, m\}$, $l \in \{1, \ldots, n\}$.*

(iii) *Let* $Y \subseteq E$. *If* $\bigvee Y$ *exists in* E *and* $x \in E$ *such that* $x \leftrightarrow y$ *for every* $y \in Y$, *then* $x \wedge (\bigvee Y) = \bigvee\{x \wedge y : y \in Y\}$ *and* $x \leftrightarrow \bigvee Y$.

A finite subset $F = (a_k)_{k=1}^n$ of effect algebra E is said to be $\oplus$-orthogonal if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ exists in E and denote $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ with $\bigoplus_{k=1}^n a_k$ or $\bigoplus F$. An arbitrary subset $G = (a_k)_{k\in H}$ of E is said to be $\oplus$-orthogonal if $\bigoplus K$ exists for every finite subset $K \subseteq G$. Let $G = (a_k)_{k\in H}$ be a $\oplus$-orthogonal subset of E. If $\bigvee\{\bigoplus K | K \subseteq G$ finite$\}$ exists in E, we denote $\bigvee\{\bigoplus K | K \subseteq G$ finite$\}$ with $\bigoplus G$ [9].

Lemma 3.2 [9] *Let* E *be a complete effect algebra and* $(a_k)_{k\in H}$ *a* $\oplus$*-orthogonal subset of* E. *If* $H_1 \subseteq H$, $H_2 = H \backslash H_1$, *then*

$$\bigoplus_{k\in H} a_k = \left(\bigoplus_{k\in H_1} a_k\right) \oplus \left(\bigoplus_{k\in H_2} a_k\right).$$

For an element x of an effect algebra E, we define $\mathrm{ord}(x) = \infty$ if $nx = x \oplus \cdots \oplus x$ (n times) exists for every $n \in N$ and $\mathrm{ord}(x) = n_x \in N$ (called isotropic index) if n_x is the greatest integer such that $n_x x$ exists in E. It is clear that in a complete lattice effect algebra E, $n_x < \infty$ for every $x \in E$.

The set of sharp elements of E is denoted by $S(E)$. It has been shown that in every lattice effect algebra E, $S(E)$ is an orthomodular lattice, is a sub-effect algebra and a sublattice of E [4]. Moreover, $S(E)$ is a full sub-lattice of E, that is, $S(E)$ inherits all suprema and infima of subsets of $S(E)$ if they exist in E. In a complete atomic distributive lattice effect algebra E, $S(E)$ is a complete atomic Boolean algebra [9].

An element $x \in E$ is said to be principle if $a \le x, b \le x$ and $a \oplus b$ is defined, then $a \oplus b \le x$.

In a lattice effect algebra E, x is principle iff x is sharp.

Lemma 3.3 [9] *If* E *is an atomic lattice effect algebra and* $a \in E$ *is an atom with* $\mathrm{ord}(a) = n_a$, *then*

(i) $(ka) \wedge (ka)' \neq 0$ *for all* $k \in \{1, 2, \ldots, n_a - 1\}$.
(ii) $n_a a \in S(E)$.
(iii) *If* $u = (k_1 a_1) \oplus (k_2 a_2) \oplus \cdots \oplus (k_n a_n)$, *where* $\{a_1, a_2, \ldots, a_n\}$ *is a set of mutually different atoms of* E, *then* $u = \bigvee_{i=1}^n (k_i a_i)$.
(iv) *If* E *is complete and* $x \neq 0$, *then there are mutually different atoms* $a_\alpha \in E, \alpha \in \Gamma$, *and positive integers* k_α *such that*

$$x = \bigoplus\{k_\alpha a_\alpha | \alpha \in \Gamma\} = \bigvee\{k_\alpha a_\alpha | \alpha \in \Gamma\}.$$

Moreover, $x \in S(E)$ *iff* $k_\alpha = n_{a_\alpha} = \mathrm{ord}(a_\alpha)$ *for all* $\alpha \in \Gamma$.

Lemma 3.4 [9] *Let* E *be a complete atomic effect algebra. Then for every* $x \in E$ *with* $x \neq 0$, *there exists a unique* $w_x \in S(E)$, *a unique set* $\{a_\alpha | \alpha \in \mathcal{A}\}$ *of atoms of* E *and unique positive integers* k_α, $\alpha \in \mathcal{A}$, *such that*

$$x = w_x \oplus \left(\bigoplus\{k_\alpha a_\alpha | \alpha \in \mathcal{A}\}\right).$$

Moreover, if $w \in S(E)$ *with* $w \le x \ominus w_x$, *then* $w = 0$.

Definition 3.1 [10–12] An effect algebra E is called sharply dominating if for every $a \in E$ there exists a smallest sharp element $\hat{a} \in E$ such that $a \leq \hat{a}$. A sharply dominating effect algebra E is called S-dominating if $a \wedge p$ exists for every $a \in E$ and $p \in S(E)$.

It is clear that a lattice effect algebra E is S-dominating iff E is sharply dominating. Every complete lattice effect algebra is sharply dominating.

Lemma 3.5 *Let E be a complete atomic distributive lattice effect algebra. If u is a finite element of E, then the smallest sharp element $\hat{u}$ dominating u is also finite. If $u_1, u_2 \in E$ with $u_1 \leq u_2$, then $\hat{u_1} \leq \hat{u_2}$.*

Proof By Lemma 3.3, there are mutually distinct atoms $\{a_1, \ldots, a_m\}$ and positive integers $\{k_1, \ldots, k_m\}$ such that $u = \bigoplus_{i=1}^m k_i a_i = \bigvee_{i=1}^m k_i a_i$. We claim $\hat{u} = \bigoplus_{i=1}^m n_i a_i = \bigvee_{i=1}^m n_i a_i$, where n_i is the isotropic index of a_i. Let $b \in S(E)$ with $u \leq b$. For every i with $k_i \neq n_i$, we have $k_i a_i \oplus b'$ is defined and since $(k_i a_i) \wedge b' \leq b \wedge b' = 0$, we obtain, by Lemma 3.1, that $(k_i a_i) \oplus b' = (k_i a_i) \vee b'$. As $k_i a_i \leq a_i'$ and $b' \leq (k_i a_i)' \leq a_i'$, $(k_i a_i) \oplus b' = (k_i a_i) \vee b' \leq a_i'$. It follows that there exists $(k_i + 1)a_i \oplus b' = (k_i + 1)a_i \vee b' \leq a_i'$. Hence, $(k_i + 2)a_i \oplus b'$ exists, by induction, $n_i a_i \oplus b'$ exists and so $n_i a_i \leq b$. Thus, $\hat{u} = \bigvee_{i=1}^m n_i a_i \leq b$.

Let $u_1 \leq u_2$. Then $u_1 \leq u_2 \leq \hat{u_2}$. By the definition of sharply dominating effect algebra, we get $\hat{u_1} \leq \hat{u_2}$. □

Lemma 3.6 *Let E be a complete atomic distributive lattice effect algebra and F the set of all finite elements and* 0 *of E. Then F is an ideal of E.*

Proof Note that in every complete atomic distributive lattice effect algebra, the join of two finite elements is finite as well, so we only need to prove the fact that if $u \in E$ is finite and $x \in E$ with $0 \neq x \leq u$, then x is finite.

Let u be finite and $0 \neq x \leq u$. Then, by Lemma 3.5, $\hat{u} \in S(E)$ and $\hat{u}$ is finite with $x \leq \hat{u}$. It follows from Lemma 2.3 that $\hat{u}$ is a compact element of E, so it is a compact element of $S(E)$.

(i) If $x \in S(E)$, by Lemma 3.3, we can assume $x = \bigoplus_{\alpha \in \Lambda} n_\alpha a_\alpha = \bigvee_{\alpha \in \Lambda} n_\alpha a_\alpha$, where $\{a_\alpha : \alpha \in \Lambda\}$ is a set of atoms and n_α is the isotropic index of a_α. Note that $S(E)$ is a complete atomic Boolean algebra and $x \leq \hat{u}$, so x is a compact element of $S(E)$. Thus, there exist $\{\alpha_1, \ldots, \alpha_m\} \subseteq \Lambda$ such that $x = \bigvee_{\alpha \in \Lambda} n_\alpha a_\alpha \leq \bigvee_{i=1}^m n_{\alpha_i} a_{\alpha_i}$, so $x = \bigvee_{i=1}^m n_{\alpha_i} a_{\alpha_i} = \bigoplus_{i=1}^m n_{\alpha_i} a_{\alpha_i}$. That is, x is a finite element of E.

(ii) If $x \notin S(E)$. There exists $x_1 \in E$ such that $\hat{u} = x \oplus x_1$. By Lemma 3.4, we can assume that

$$x = w_x \oplus \left(\bigvee_{\alpha \in \Lambda} k_\alpha b_\alpha \right) = w_x \oplus \left(\bigoplus_{\alpha \in \Lambda} k_\alpha b_\alpha \right),$$

$$x_1 = w_{x_1} \oplus \left(\bigvee_{\beta \in \Gamma} l_\beta c_\beta \right) = w_{x_1} \oplus \left(\bigoplus_{\beta \in \Gamma} l_\beta c_\beta \right),$$

where $w_x, w_{x_1} \in S(E)$, $\{b_\alpha : \alpha \in \Lambda\}$ and $\{c_\beta : \beta \in \Gamma\}$ are sets of atoms and $k_\alpha \neq n_\alpha$, $l_\beta \neq n_\beta$ for every $\alpha \in \Lambda$ and $\beta \in \Gamma$. Note that $S(E)$ is a sub-effect algebra, denote $x_0 = (\bigoplus_{\alpha \in \Lambda} k_\alpha b_\alpha) \oplus (\bigoplus_{\beta \in \Gamma} l_\beta c_\beta)$, we obtain $x_0 = \hat{u} \ominus w_x \ominus w_{x_1} \in S(E)$. Denote $\Lambda_1 = \{\alpha \in \Lambda :$ there exists c_β such that $b_\alpha = c_\beta\}$ and $\Gamma_1 = \{\beta \in \Gamma :$ there exists b_α such that $c_\beta = b_\alpha\}$. For every $\beta \in \Gamma_1$, if $c_\beta = b_\alpha$, then we denote $l_\alpha = l_\beta$. Thus, by Lemma 3.2,

$$\left(\bigoplus_{\alpha\in\Lambda} k_\alpha b_\alpha\right) \oplus \left(\bigoplus_{\beta\in\Gamma} l_\beta c_\beta\right) = \bigoplus_{\alpha\in\Lambda_1} (k_\alpha + l_\alpha) b_\alpha \oplus \left(\bigoplus_{\alpha\in\Lambda\setminus\Lambda_1} k_\alpha b_\alpha\right) \oplus \left(\bigoplus_{\beta\in\Gamma\setminus\Gamma_1} l_\beta c_\beta\right).$$

So $b_\alpha \neq c_\beta$ for every $\alpha \in (\Lambda \setminus \Lambda_1)$ and $\beta \in (\Gamma \setminus \Gamma_1)$. It follows from Lemma 3.1 that $(k_\alpha b_\alpha) \wedge (l_\beta c_\beta) = 0$ for every $\alpha \in (\Lambda \setminus \Lambda_1)$ and $\beta \in (\Gamma \setminus \Gamma_1)$. As $(k_\alpha b_\alpha) \oplus (l_\beta c_\beta)$ is defined, $(k_\alpha b_\alpha) \leftrightarrow (l_\beta c_\beta)$, where $\alpha \in (\Lambda \setminus \Lambda_1)$ and $\beta \in (\Gamma \setminus \Gamma_1)$. By Lemma 3.1, we have

$$k_\alpha b_\alpha \leftrightarrow \bigvee_{\beta\in\Gamma\setminus\Gamma_1} (l_\beta c_\beta), \qquad \bigvee_{\alpha\in\Lambda\setminus\Lambda_1} (k_\alpha b_\alpha) \leftrightarrow \bigvee_{\beta\in\Gamma\setminus\Gamma_1} (l_\beta c_\beta).$$

So

$$\left(\bigvee_{\alpha\in\Lambda\setminus\Lambda_1} (k_\alpha b_\alpha)\right) \wedge \left(\bigvee_{\beta\in\Gamma\setminus\Gamma_1} (l_\beta c_\beta)\right) = \bigvee_{\alpha\in\Lambda\setminus\Lambda_1, \beta\in\Gamma\setminus\Gamma_1} ((k_\alpha b_\alpha) \wedge (l_\beta c_\beta)) = 0.$$

Hence

$$\left(\bigvee_{\alpha\in\Lambda\setminus\Lambda_1} (k_\alpha b_\alpha)\right) \oplus \left(\bigvee_{\beta\in\Gamma\setminus\Gamma_1} (l_\beta c_\beta)\right) = \left(\bigvee_{\alpha\in\Lambda\setminus\Lambda_1} (k_\alpha b_\alpha)\right) \vee \left(\bigvee_{\beta\in\Gamma\setminus\Gamma_1} (l_\beta c_\beta)\right).$$

That is

$$\left(\bigoplus_{\alpha\in\Lambda\setminus\Lambda_1} k_\alpha b_\alpha\right) \oplus \left(\bigoplus_{\beta\in\Gamma\setminus\Gamma_1} l_\beta c_\beta\right) = \left(\bigvee_{\alpha\in\Lambda\setminus\Lambda_1} (k_\alpha b_\alpha)\right) \vee \left(\bigvee_{\beta\in\Gamma\setminus\Gamma_1} (l_\beta c_\beta)\right).$$

Similarly, we can prove $x_0 = (\bigvee_{\alpha\in\Lambda_1} (k_\alpha + l_\alpha) b_\alpha) \vee (\bigvee_{\alpha\in\Lambda\setminus\Lambda_1} k_\alpha b_\alpha) \vee (\bigvee_{\beta\in\Gamma\setminus\Gamma_1} l_\beta c_\beta)$. For every fixed $\alpha_0 \in \Lambda \setminus \Lambda_1$, $(k_{\alpha_0} a_{\alpha_0}) \wedge x_0' \le x_0 \wedge x_0' = 0$. As $\bigoplus_{\alpha\in\Lambda_1} (k_\alpha + l_\alpha) b_\alpha \oplus (\bigoplus_{\alpha\in\Lambda\setminus\Lambda_1} k_\alpha b_\alpha) \oplus (\bigoplus_{\beta\in\Gamma\setminus\Gamma_1} l_\beta c_\beta)$ is defined, we have

$$k_{\alpha_0} a_{\alpha_0} \le \left(\bigvee_{\alpha\in\Lambda_1} (k_\alpha + l_\alpha) b_\alpha\right)', \qquad k_{\alpha_0} a_{\alpha_0} \le \left(\bigvee_{\alpha\in\Lambda\setminus\Lambda_1, \alpha\neq\alpha_0} k_\alpha b_\alpha\right)',$$

$$k_{\alpha_0} a_{\alpha_0} \le \left(\bigvee_{\beta\in\Gamma\setminus\Gamma_1} l_\beta c_\beta\right)'.$$

So $(k_{\alpha_0} a_{\alpha_0}) \wedge x_0' = (k_{\alpha_0} a_{\alpha_0}) \wedge ((\bigvee_{\alpha\in\Lambda_1} (k_\alpha + l_\alpha) b_\alpha) \vee (\bigvee_{\alpha\in\Lambda\setminus\Lambda_1} k_\alpha b_\alpha) \vee (\bigvee_{\beta\in\Gamma\setminus\Gamma_1} l_\beta c_\beta))' = (k_{\alpha_0} a_{\alpha_0}) \wedge (k_{\alpha_0} a_{\alpha_0})' = 0$. By Lemma 3.3, we have $k_{\alpha_0} = n_{\alpha_0}$. Note that we have assumed that $k_\alpha \neq n_\alpha$ for every $\alpha \in \Lambda$, so $\Lambda = \Lambda_1$. Similarly, we have $\Gamma = \Gamma_1$ and $k_\alpha + l_\alpha = n_\alpha$ for every $\alpha \in \Lambda$. That is

$$\left(\bigoplus_{\alpha\in\Lambda} k_\alpha b_\alpha\right) \oplus \left(\bigoplus_{\beta\in\Gamma} l_\beta c_\beta\right) = \bigvee_{\alpha\in\Lambda} (k_\alpha + l_\alpha) b_\alpha = \bigvee_{\alpha\in\Lambda} n_\alpha b_\alpha = \hat{u} \ominus w_x \ominus w_{x_1} \in S(E).$$

Since $\bigvee_{\alpha\in\Lambda} n_\alpha b_\alpha \le \hat{u}$, $\hat{u}$ is a compact element of $S(E)$ and $S(E)$ is a complete atomic Boolean algebra, we get $\bigvee_{\alpha\in\Lambda} n_\alpha b_\alpha$ is compact. Thus, there exists $\{\alpha_1, \dots, \alpha_m\} \subseteq \Lambda$ such that

$$\bigvee_{\alpha\in\Lambda} n_\alpha b_\alpha = \bigvee_{i=1}^{m} n_{\alpha_i} b_{\alpha_i} = \bigoplus_{i=1}^{m} n_{\alpha_i} b_{\alpha_i}.$$

As $\bigvee_{\alpha\in\Lambda} n_\alpha b_\alpha = \bigoplus_{\alpha\in\Lambda} n_\alpha b_\alpha = \bigoplus_{i=1}^{m} n_{\alpha_i} b_{\alpha_i}$, Λ is finite. Therefore $x = w_x \oplus (\bigvee_{\alpha\in\Lambda} k_\alpha b_\alpha)$ is finite. The lemma is proved. □

Recall that the interval topology τ_i of an effect algebra E is the topology which is defined by taking all closed interval $[a, b]$ as a sub-basis of closed sets of E. It is well known that $\tau_{id} \geq \tau_i$ and $\tau_o \geq \tau_i$ in every lattice.

Lemma 3.7 [2] *Let E be a complete atomic distributive lattice effect algebra. Then its interval topology τ_i is compact Hausdorff topology and $\tau_o = \tau_i$.*

Theorem 3.1 *Let E be a complete atomic distributive lattice effect algebra. Then τ_{id} is Hausdorff and $\tau_{id} \geq \tau_o$. Moreover, the following conditions are equivalent*:

(i) $\tau_{id} = \tau_o$.
(ii) 1 *is finite.*
(iii) *Every element of E is finite.*
(iv) *τ_{id} and τ_o are both discrete topologies.*

Proof It follows from Lemma 3.7 and $\tau_{id} \geq \tau_i$ that τ_{id} is Hausdorff topology and $\tau_{id} \geq \tau_o$.

(ii) ⇔ (iii) can be proved by Lemma 3.6 easily.

(iii) ⇒ (iv). For every $x \in E$, x and x' are both finite. By Lemma 2.4, $[0, x]$ and $[x, 1]$ are τ_o-open, so $\{x\} = [0, x] \cap [x, 1]$ is τ_o open, this showed that τ_o is discrete. Note that $\tau_{id} \geq \tau_o$, we have τ_{id} and τ_o are both discrete.

(iv) ⇒ (i) is obvious.

(i) ⇒ (ii). Assume 1 is not finite. Let F_0 be the set of all finite elements and 0. Then it follows from Lemma 3.6 that F_0 is an ideal and $1 \notin F_0$. By the Zorn's Lemma, F_0 is contained in an ideal F maximal subject to not containing 1. It is easy to prove that $F' = \{f' \in E : f \in F\}$ is a maximal dual ideal and so τ_{id}-open with $1 \in F'$. As $1 = \vee\{u \in E : u$ is finite$\}$, we can choose a net of finite elements $(u_\alpha)_{\alpha\in\Lambda}$ of E such that $u_\alpha \uparrow 1$, thus, by Lemma 3.5, $(\hat{u}_\alpha)_{\alpha\in\Lambda}$ is also a net of finite elements of E and $\hat{u}_\alpha \uparrow 1$. Note that $(u_\alpha)_{\alpha\in\Lambda} \subseteq F$ and $\hat{u}_\alpha \vee \hat{u}'_\alpha = 1$, we have $\hat{u}'_\alpha \notin F$, otherwise $1 \in F$. Hence $\hat{u}_\alpha \notin F'$. That is, $\hat{u}_\alpha$ is not τ_{id}-convergent to 1. However, $\hat{u}_\alpha \xrightarrow{(\tau_o)} 1$. This contradicts (i). So 1 is finite. □

4 The Order Topology Continuity of Operation ⊕ of Effect Algebras

For the order topology continuity of operation ⊕, Wu had only presented a sufficient condition under a very strictly assumption [13]. Now, we study this question continuously.

Lemma 4.1 [14] *Let (X, T) be a topology space. Then (X, T) is Hausdorff space iff every convergent net in X has exactly one limit point.*

Lemma 4.2 [2] *Let E be a complete (o)-continuous lattice effect algebra, $x_\alpha, x, y \in E$. Then*

(i) $x_\alpha \xrightarrow{(\tau_o)} x \Rightarrow x_\alpha \vee y \xrightarrow{(\tau_o)} x \vee y$.
(ii) $x_\alpha \xrightarrow{(\tau_o)} x \Rightarrow x_\alpha \wedge y \xrightarrow{(\tau_o)} x \wedge y$.
(iii) $x_\alpha \xrightarrow{(\tau_o)} x \Rightarrow x'_\alpha \xrightarrow{(\tau_o)} x'$.

Our main results in this section are the following:

Theorem 4.1 *Let* $(E, \oplus, 0, 1)$ *be a complete* (o)*-continuous lattice effect algebra. If* $\oplus$ *is order topology* τ_o *continuous, then* τ_o *is Hausdorff topology.*

Proof Let $(x_\alpha)_{\alpha\in\Lambda}$ be a net in E and $x_\alpha \xrightarrow{(\tau_o)} x$, $x_\alpha \xrightarrow{(\tau_o)} y$. By Lemma 4.2,

$$x_\alpha \vee x \xrightarrow{(\tau_o)} x,\ x_\alpha \vee x \xrightarrow{(\tau_o)} x \vee y,\ (x_\alpha \vee x)' \xrightarrow{(\tau_o)} (x \vee y)'.$$

It follows from $\oplus$ is τ_o continuous that

$$(x_\alpha \vee x) \oplus (x_\alpha \vee x)' \xrightarrow{(\tau_o)} x \oplus (x \vee y)'.$$

That is, $1 \xrightarrow{(\tau_o)} x \oplus (x \vee y)'$. Note that $\tau_o \geq \tau_i$, $1 \xrightarrow{(\tau_i)} x \oplus (x \vee y)'$ and $\{1\}$ is τ_i-closed, we have $x \oplus (x \vee y)' = 1$, so $x = x \vee y$, $y \leq x$. Similarly, we can prove that $x \leq y$. Thus $x = y$ and τ_o is Hausdorff topology. □

Example 4.1 Let L be the complete Boolean algebra of all regular open subsets of the unit interval I. It follows from L is (o)-continuous and the order topology τ_o of L is not Hausdorff topology [15] and Theorem 4.1 that $\oplus$ is not order topology τ_o continuous.

Theorem 4.2 *Let* $(E, \oplus, 0, 1)$ *be a complete atomic* (o)*-continuous lattice effect algebra. If* $a_\alpha \xrightarrow{(\tau_o)} 0$ *and* $b_\alpha \xrightarrow{(\tau_o)} 0$ *with* $a_\alpha \leq b'_\alpha$ *for every* α, *then* $a_\alpha \oplus b_\alpha \xrightarrow{(\tau_o)} 0$.

Proof Suppose $\wedge_\beta \bigvee_{\alpha\geq\beta}(a_\alpha \oplus b_\alpha) = c$. For every finite element $u \in E$, there exists sharply dominating element $\hat{u}$ such that $u \leq \hat{u}$. It follows from Lemma 3.5 that $\hat{u}$ is also finite. So $[0, \hat{u}'] \subseteq [0, u']$ and by Lemma 2.4 that they are both τ_o-open. Note that $0 \in [0, \hat{u}']$, there exists α_0 such that for every $\alpha \geq \alpha_0$, $a_\alpha \in [0, \hat{u}']$ and $b_\alpha \in [0, \hat{u}']$. Thus, for every $\alpha \geq \alpha_0$, $a_\alpha \oplus b_\alpha \leq \hat{u}'$ since $\hat{u}'$ is principle, so $\bigvee_{\alpha\geq\alpha_0}(a_\alpha \oplus b_\alpha) \leq \hat{u}'$ and $c \leq \hat{u}'$, this showed that $c' \geq \hat{u} \geq u$. As $1 = \vee\{u \in E : u \text{ is finite }\}$, $c' \geq 1$. That is, $c' = 1$ and $c = 0$. Hence, we have $a_\alpha \oplus b_\alpha \xrightarrow{(o)} 0$, thus, $a_\alpha \oplus b_\alpha \xrightarrow{(\tau_o)} 0$. □

Lemma 4.3 [16] *Let* E *be a lattice effect algebra. Then* $x_\alpha \xrightarrow{(o)} x$ *iff* $(x \vee x_\alpha) \ominus x \xrightarrow{(o)} 0$ *and* $x \ominus (x \wedge x_\alpha) \xrightarrow{(o)} 0$.

Theorem 4.3 *Let* $(E, \oplus, 0, 1)$ *be a complete atomic* (o)*-continuous lattice effect algebra. If* $a_\alpha \xrightarrow{(\tau_o)} a$ *and* $b_\alpha \xrightarrow{(\tau_o)} b$ *with* $a_\alpha \leq b'_\alpha$ *and* $a_\alpha \leq b'$ *and* $b_\alpha \leq a'$ *for every* α, *then* $a_\alpha \oplus b_\alpha \xrightarrow{(\tau_o)} a \oplus b$.

Proof By Theorem 2.1 and Lemma 4.3, we only need to prove that

$$((a_\alpha \oplus b_\alpha) \vee (a \oplus b)) \ominus (a \oplus b) \xrightarrow{(\tau_o)} 0,\ (a \oplus b) \ominus ((a \oplus b) \wedge (a_\alpha \oplus b_\alpha)) \xrightarrow{(\tau_o)} 0.$$

Note that $a_\alpha \xrightarrow{(\tau_o)} a$ implies that $a_\alpha \xrightarrow{(\tau_i)} a$, and since $a_\alpha \leq b'$ for every α, we have $a \leq b'$. Moreover, since $a_\alpha \leq (b'_\alpha \wedge b')$ and $a \leq (b'_\alpha \wedge b')$, $(a_\alpha \vee a) \oplus (b_\alpha \vee b)$ is defined. As $a_\alpha \xrightarrow{(\tau_o)} a$

and $b_\alpha \xrightarrow{(\tau_o)} b$, it follows from Lemma 4.3 that

$$(a_\alpha \vee a) \ominus a \xrightarrow{(\tau_o)} 0, \ (b_\alpha \vee b) \ominus b \xrightarrow{(\tau_o)} 0.$$

Note that $((a_\alpha \vee a) \ominus a) \oplus ((b_\alpha \vee b) \ominus b) = ((a_\alpha \vee a) \oplus (b_\alpha \vee b)) \ominus (a \oplus b) \geq ((a_\alpha \oplus b_\alpha) \vee (a \oplus b)) \ominus (a \oplus b)$ and Theorem 4.2, we have

$$((a_\alpha \vee a) \ominus a) \oplus ((b_\alpha \vee b) \ominus b) \xrightarrow{(\tau_o)} 0,$$

so

$$((a_\alpha \oplus b_\alpha) \vee (a \oplus b)) \ominus (a \oplus b) \xrightarrow{(\tau_o)} 0.$$

Similarly, we can prove that $(a \oplus b) \ominus ((a \oplus b) \wedge (a_\alpha \oplus b_\alpha)) \xrightarrow{(\tau_o)} 0$ and the theorem is proved. □

Corollary 4.1 *Let $(E, \oplus, 0, 1)$ be a complete atomic MV-effect algebra, $a_\alpha \xrightarrow{(\tau_o)} a$ and $b_\alpha \xrightarrow{(\tau_o)} b$ with $a_\alpha \leq b'_\alpha$ for every α. Then $a_\alpha \oplus b_\alpha \xrightarrow{(\tau_o)} a \oplus b$.*

Acknowledgements The authors wish to express their thanks to the referees for their valuable comments and suggestions.

References

1. Foulis, D., Bennett, M.K.: Effect algebras and unsharp quantum logics. Found. Phys. **24**, 1331–1352 (1994)
2. Riečanová, Z.: Order-topological lattice effect algebras. In: Contributions to General Algebra 15. Proceeding of the Klagenfurt Workshop 2003 on General Algebra, Klagenfurt, Austria, pp. 151–159 (2003)
3. Erné, M., Riečanová, Z.: Order-topological complete orthomodular lattices. Topol. Appl. **37**, 215–227 (1995)
4. Riečanová, Z.: Smearings of states defined on sharp elements onto effect algebras. Int. J. Theor. Phys. **41**, 1511–1524 (2002)
5. Atherton, C.R.: Concerning intrinsic topologies on Boolean algebras and certain bicompactly generated lattices. Glasg. Math. J. **11**, 156–161 (1970)
6. Ward, A.J.: On relations between certain intrinsic topologies in partially ordered sets. Proc. Camb. Philos. Soc. **51**, 254–261 (1955)
7. Frink, O.: Ideals in partially ordered sets. Am. Math. Mon. **61**, 2319–2323 (1954)
8. Kôpka, F.: Compatibility in D-posets. Int. J. Theor. Phys. **34**, 1525–1531 (1995)
9. Riečanová, Z.: Basic decomposition of elements and Jauch-Piron effect algebras. Fuzzy Sets Syst. **155**, 138–149 (2005)
10. Gudder, S.P.: Sharply dominating effect algebras. Found. Phys. **24**, 1331–1352 (1994)
11. Gudder, S.P.: S-dominating effect algebra. Int. J. Theor. Phys. **37**, 1525–1531 (1998)
12. Riečanová, Z., Wu, J.D.: States on sharply dominating effect algebras. Sci. China (A) **51**, 907–914 (2008)
13. Wu, J.D., Tang, Z.F., Cho, M.H.: Two variables operation continuity of effect algebras. Int. Theor. Phys. **44**, 581–586 (2005)
14. Wilansky, A.: Topology for Analysis. Ginn, Waltham (1970)
15. Floyd, E.E.: Boolean algebras with pathological order topologies. Pac. J. Math. **5**, 687–699 (1955)
16. Riečanová, Z.: States, uniformities and metrics on lattice effect algebras. Int. J. Uncertain. Fuzziness, Knowl. Based Syst. **10**, 125–133 (2002)